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Probing Spacetime: A Review of Causal Set Theory

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Abstract

The causal set approach to quantum gravity posits spacetime to be fundamentally discrete and brings its causal structure to the forefront. This review outlines the physics behind causal sets and identifies open problems in their kinematics, dynamics and phenomenology.

1 Introduction

Physicists currently find themselves at a crossroad on the journey towards unification. Einstein's General theory of Relativity (GR), which deals with the structure of spacetime and the nature of gravity, is formulated based on classical physics. In contrast, non-gravitational matter is understood within the framework of quantum mechanics (QM). As a result, there are numerous incompatibilities between the two, including the fact that QM is fundamentally stochastic, unlike deterministic GR [1]. In order to marry these two theories into a single theory of quantum gravity one must first decide which concepts, whether old or new, are to be the fundamental concepts of the unified theory.

Causal set theory (CST) approaches quantum gravity by keeping the concept of causal order from GR and the path integral from QM. It then leaps into new physical territory by doing away with the continuous manifold of GR and instead hypothesizing spacetime to be fundamentally discrete [2].

2 Why Causal Sets?

During the final quarter of the 20th century, the notion of a discrete spacetime, together with the importance of causal order, was hypothesized independently by various physicists [3–5]. In fact, even Einstein suspected spacetime to be discrete [6], however he did not expound the topic because, as he said[1], "we lack the mathematical structure unfortunately". Perhaps this mathematical structure is provided by causal sets, whose motivations will now be discussed.

2.1 The Infinities

There are multiple quantities in modern physics that diverge to infinity, all of which could be avoided if a shortest physical length cutoff is applied – a natural consequence of discrete spacetime. These include amplitudes in quantum field theory (QFT)[2], the spacetime curvature of singularities and the entanglement entropy of a black hole. The third of these is the most enlightening because removing this infinity *requires* a wavelength cutoff on the quantum modes covering the black hole horizon [8]. Hence, given that the entropy of a black hole scales as its horizon area (in Planckian

[1] Translated by [7].
[2] Although renormalization can remove this infinity in a flat spacetime, it fails in a curved spacetime [1].

units) [9], it must be that there is a length cutoff. Moreover, it implies that the discreteness scale is of order the Planck length, $\sim 10^{-35}$m, far shorter than anything already measured [1].

2.2 Fundamental Causal Structure

Causal order is a key part of GR. Every event in a spacetime has a causal future and past; every event can only influence, or be influenced by, specific other events. For two points $x, y \in \mathcal{M}$ in a spacetime $(\mathcal{M}, g)$, where $\mathcal{M}$ is a Lorentzian manifold endowed with a metric g, $x \leq y$ is an order relation implying that y is in the causal future of x [2].

Work in global causal analysis [10–15] has shown that knowing the causal structure of a distinguishing spacetime[3], that is knowing the order relations between all points, is enough to determine the geometrical properties such as topology, differentiable structure and the metric up to a conformal factor. Therefore, in a continuum theory, causal structure alone is not sufficient to determine the full geometry, as independent knowledge of the volume element to compute the conformal factor is also needed [2].

However, as alluded to by Riemann before the advent of CST, the volume of a region of a "discrete manifold" can be computed by counting the elements corresponding to that region [16]. Volume information is encoded in the discretum and thus causal order alone is enough to determine the full geometry of the spacetime; in the words of Sorkin, "Number + Order = Geometry" [17]. This unification provided by causal structure is a major reason to treat it as fundamental.

Further motivation [2] comes from attempts at unifying black hole thermodynamics, in which causal order is a key ingredient in generalising the second law [18]. Also, the Lorentzian signature of spacetime $(D + 1)$ is a natural consequence of CST, as it is the only signature with a notion of past and future [19, 20].

3 What is a Causal Set?

A causal set (causet) is a locally finite partially ordered set. Mathematically, it is a set $\mathcal{C}$ together with a binary relation $\preceq$ satisfying the following axioms:

(i) $x \preceq y \preceq z \Rightarrow x \preceq z, \forall\ x, y, z \in \mathcal{C}$ (transitivity);

(ii) $x \preceq y$ and $y \preceq x \Rightarrow x = y, \forall\ x, y \in \mathcal{C}$ (acyclicity);

(iii) $|\{y \in \mathcal{C} | x \preceq y \preceq z\}| < \infty, \forall\ x, z \in \mathcal{C}$ (local finiteness).

In words, $x \prec y$ means "x causally precedes y".[4] Axioms (i) and (ii) imply that the set is ordered. This is only a partial order as two elements corresponding to events with a spacelike separation are not related to each other. Axiom (iii) ensures discreteness [2].

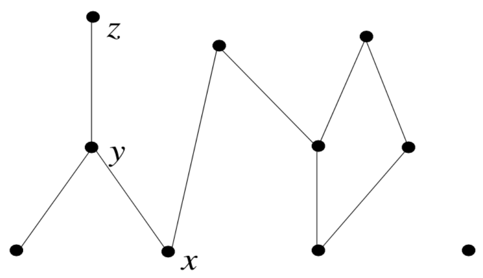

Figure 1: A Hasse diagram. Links are only drawn between nearest neighbours, thus there is no line directly from x to z. [21]

[3] A distinguishing spacetime is one in which each point has a unique chronological past and future [2].

[4] This review uses the convention that $x \prec y$ implies that x precedes y but $x \neq y$, whereas $\preceq$ is reflexive.

Figure 1 depicts a visualisation of a causet, known as a Hasse diagram [21]. This is akin to an upside down family tree, following the convention that the time axis points upwards in relativity. Each node represents an element and the *links* represent causal connections between nearest neighbours, where x and y are nearest neighbours iff $|\{v \in \mathcal{C}|x \prec v \prec y\}| = 0$. The diagram contains no notion of length, reflecting the fact that a causet has no inbuilt length information – it is simply a set of elements with order relations.

4 Continuum Correspondence

According to CST, the continuum does not exists. It is merely an approximation to a causet at scales much larger than the discreteness scale.

In order to obtain a causet for which a spacetime $(\mathcal{M}, g)$ is a good approximation, *sprinkling* is performed. Points in a spacetime are selected at random according to a Poisson process

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!} \tag{1}$$

where ρ is the density of sprinkling and n is the number of points sprinkled in a volume V. A Poisson distribution is used both to respect Lorentz invariance and to ensure that, on average, the number of selected points within some region of the spacetime is proportional to the volume of that region. The set of these selected points, together with their causal order, forms a causet which is manifoldlike, or more rigorously, *faithfully embeds* into $(\mathcal{M}, g)$ [21].

The "hauptvermutung", or "fundamental conjecture", of CST states that a single causet being faithfully embeddable into multiple manifolds necessarily implies that the manifolds are *similar* on large scales. If falsified, there would be ambiguity as to which spacetime a causet corresponds to and troubling questions such as why we only experience one spacetime would arise. It has been proven true in the $\rho \to \infty$ case [22] and some progress has been made towards a general proof [23–25], although the lack of a rigorous definition of the word *similar* makes such a proof complicated to construct [19, 21].

If causets are all that there is, there should be a method of going directly from causet to manifold, unlike sprinkling which goes the other way. This is especially important given the entropy problem of CST which concerns the fact that numerous causets are non-manifoldlike [21], see for example Kleitman-Rothschild causets [26]. Work towards finding a necessary and sufficient method of determining whether a causet is manifoldlike, and if so what the manifold is, is being undertaken [21]. Some computational methods of achieving this in small regions of Minkowski spacetime [27] and some necessary conditions [28] have already been found. Furthermore, work in the combinatorial theory of partial orders has produced ways of calculating properties of the continuum from the causet substructure [29], including the dimensionality of spacetime [3,19,30,31], timelike distances [19, 32, 33], spacelike distances [34], and scalar curvature [35, 36]. Much of this only applies in special cases and so is a work in progress [19].

5 Nonlocality

Enforcing Lorentz invariance on a discrete spacetime leads to nonlocality. A heuristic explanation for this can be obtained by considering a sprinkling onto 1+1D Minkowski spacetime in which the surface of constant proper time is a hyperbola. Assuming links are of order one Planck unit of proper time, they will connect to points lying over the infinite extent of the hyperbola. This represents nonlocality in the sense that an event can directly affect another event which is very far away and much later in time. This was an immediate concern of Moore [37], but resolutions have been since suggested by Bombelli et al [38], including using a dynamics that does not produce such nonlocal links on a macroscopic scale, i.e. nonlocality only exists up to some intermediate scale.

Note that Lorentz invariance has been verified to much lower scales than locality, hence the choice of sacrificing locality [21]. However, it is possible that Lorentz invariance breaks down too.

These infinite nearest neighbours make it difficult to define discrete differential operators because there is no simple analogue of the finite difference methods employed in Euclidean space [21].

6 Dynamics

In order to address the question of how causets grow, a discrete analogue of the sum-over-histories (SOH) approach to QM is employed. SOH is chosen because it is fundamentally a spacetime approach, unlike the wavefunction approach which takes place in space alone and struggles with its interpretation of time [39].

It was initially proposed [5, 29] to take an approach analogous to the Feynman path integral, where each causet is assigned some amplitude $e^{iS(\mathcal{C})}$ given its action $S(\mathcal{C})$. Such an action would be expected to correspond to the Einstein-Hilbert action, which is an integral of a local quantity [21]. Thus the problem of nonlocality brought difficulties in defining such a causet action, causing the method to stall. Recent work on discretising the d'Alembertian [35, 36, 40, 41] has shown that the action can be defined based on this d'Alembertian and some prefactor, giving this technique renewed hope.

Nevertheless, an alternative approach to dynamics, know as sequential growth (SG), has become popular. Classical SG [42] is a stochastic (Markovian) process in which the dynamics is described by a series of transitions. A single new element is added to a causet in each transition, with its ancestors determined probabilistically. This method can be understood graphically in figure 2.

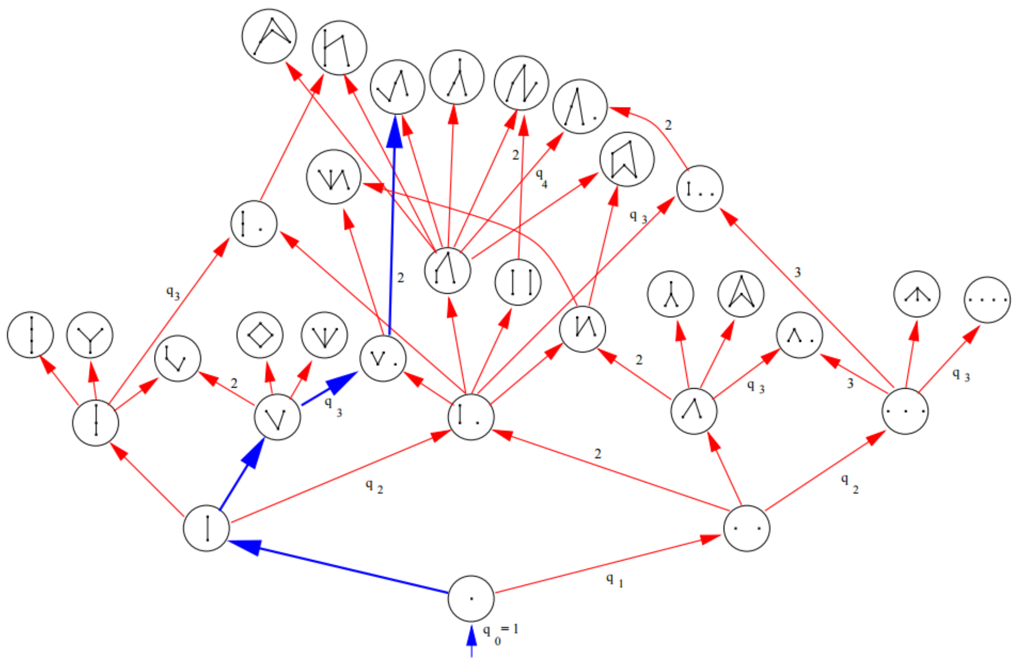

Figure 2: A Hasse diagram of Hasse diagrams showing possible initial evolutions of a causet of 0 cardinality via SG. The arrows represent transitions, each of which have some *transition probability* of occurring. The blue arrows show one particular growth. [42]

In order to determine the laws of dynamics, physical constraints must be applied:

(i) **Internal temporality**: An element cannot be born before its ancestors;

(ii) **General covariance**: The probability of growing from one causet to another is independent of the order in which the elements are added;

(iii) **Bell causality**: An element being born in one part of the causet cannot affect a spacelike part of the causet.

Applying these constraints yields the "Rideout–Sorkin" class of solutions, which are transitive percolations. They avoid the entropy problem by assigning low probabilities to non-manifoldlike causets [42].

On a philosophical level, SG makes the notion of *temporal becoming* seem natural as the element by element growth resembles the passing of time [39, 43]. This is unlike the block picture of spacetime where, in some sense, the entirety of time exists 'now'. Some metaphysical implications of time within SG include the fact that transitive percolation is time reversal invariant, allowing for a sense of becoming both into the future and past; and the past being indeterminate, changing as a causet grows [44].

The end goal is to achieve a quantum version of SG. This seems most readily achievable by replacing the transition probabilities with transition amplitudes, which could then be used to calculate the quantum measure. However, defining the Bell causality constraint in terms of quantum measure theory [45, 46] is proving difficult [42, 47], although some progress has been made [48]. Nonetheless, one attempt at quantum SG [49] introduces a complex percolation process, obeying causality and covariance.

7 Phenomenology

Accurate phenomenology requires a better understanding of dynamics and particle propagation on a causet. Some progress in doing QFT on a fixed background causet has been made [50, 51]. One approach uses a discrete form of the path integral to compute the particle propagator and a matrix geometric series to calculate the sum-over-trajectories; so far this method only applies to scalar particles. Nevertheless, due to the concrete kinematics of CST it is possible to form heuristic phenomenology without dwelling on the dynamics.

Some examples of phenomenology include the diffusion and drift of photon polarisation, however this has yet to be observed in CMB data [52]; investigating the correction terms produced by the discretised Klein-Gordon propagator [53, 54] and d'Alembertian [55]; and the fact that generic SG models lead to cycles of big bangs and crunches, after each of which the free parameters of the model are renormalised and converge towards a small set of values which could explain apparent fine tuning [56–58].

7.1 Cosmological Constant

The most successful piece of CST phenomenology suggests that the cosmological 'constant' Λ is not actually constant. The standard deviation of the volume V of a region of spacetime can be shown to be $\delta V = \sqrt{V}$ by using equation (1) and the fact 'number equals volume'. Assuming a fixed causet cardinality (unimodular gravity), the action-integral contains a term of the form ΛV in the classical limit. This means that Λ and V are conjugate variables and thus $\delta\Lambda\, \delta V \sim 1$. Using the current volume of the observable universe gives $\delta\Lambda \sim 1/\sqrt{V} \sim 10^{-120}$. If one assumes that, for some currently unknown reason, the mean cosmological constant is 0, then this fluctuation agrees with cosmological data [19].

More sophisticated models, capable of determining the correlation between fluctuations at different times, would be useful further work as it would enable comparison with constraints on Λ at various cosmological epochs [19]. This has been attempted in [59].

7.2 Swerves

It was found that, under a simple model of particle propagation, particles swerve away from their classical geodesic path. In this model, a particle occupies one causet element, of a sprinkling, at a time and is made to move along its geodesic as much as discreteness allows. Nonlocality is bypassed by restricting particles to move between elements within some 'forgetting proper time' of one and other [60].

This model failed at providing an alternative explanation to "Fermi acceleration" [61] for the source of high energy cosmic rays, due to laboratory constraints preventing sufficiently high accelerations from being possible [60]. There are other proposed astrophysical phenomenon, for example involving high energy neutrinos [62], that could potentially be explained by swerves. Alternatively, perhaps the swerves could be directly observed by a particle detector [60]. Looking forward, a quantum model would be a big step towards revealing more promising results [1].

8 Conclusion

Various aspects of CST have been discussed, however there exists far more depth than could be included here. For example, the concept of coarse graining, which provides a different notion of causet-continuum correspondence [29]. Multiple areas for further work have been identified, including recovering locality and developing phenomenology.

In 1905 Einstein revealed matter to be discrete by explaining the phenomenon of Brownian motion [63] – a monumental discovery at the time. In 1916 he went on to publish his paper on GR and with it the idea of a continuous spacetime, although he questioned if it was truly continuous. One century on, the baton lies with us as we seek to continue Einstein's legacy and in turn unify physics.

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